Solutions for Bayesian networks and decision graphs (second edition)

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Solution for exercise 1.1

Define $\mathcal{B}' = \mathcal{B} \setminus (\mathcal{A} \cap \mathcal{B})$. We immediately have

$$\mathcal{A} \cap \mathcal{B}' = \emptyset \; , \tag{1}$$

$$\mathcal{A} \cup \mathcal{B}' = \mathcal{A} \cup \mathcal{B} \; , \tag{2}$$

and

$$(\mathcal{A} \cap \mathcal{B}) \cup \mathcal{B}' = \mathcal{B} \; . \tag{3}$$

From (1), we have

$$(\mathcal{A} \cap \mathcal{B}) \cap \mathcal{B}' = \emptyset \; . \tag{4}$$

From (3), Axiom 3, and (4) we have

$$\begin{aligned} & P(\mathcal{B}) = \mathcal{P}((\mathcal{A} \cap \mathcal{B}) \cup \mathcal{B}') = \mathcal{P}(\mathcal{A} \cap \mathcal{B}) + \mathcal{P}(\mathcal{B}') \\ \Leftrightarrow & P(\mathcal{B}') = \mathcal{P}(\mathcal{B}) - \mathcal{P}(\mathcal{A} \cap \mathcal{B}) \; . \end{aligned} \tag{5}$$

From (2), Axiom 3, and (1) we have

$$P(\mathcal{A} \cup \mathcal{B}) = \mathcal{P}(\mathcal{A} \cup \mathcal{B}') = \mathcal{P}(\mathcal{A}) + \mathcal{P}(\mathcal{B}') \; . \tag{6}$$

The result now follows from (5) and (6).

Solution for exercise 1.2

Sample space (r stands for "red", and b for "blue"):

$$\mathcal{S}_1 = \{r1b1, r1b2, \ldots, r1b6, r2b1, r2b2, \ldots, r6b6\} \; .$$

Each element has probability $\frac{1}{36}$.

Sample space two: $\mathcal{S}_2 = \{2, \ldots, 12\}$ with probabilities $P(7) = \frac{6}{36}$, $P(6) = P(8) = \frac{5}{36}$, $P(5) = P(9) = \frac{4}{36}$, $P(4) = P(10) = \frac{3}{36}$, $P(3) = P(11) = \frac{2}{36}$, and $P(2) = P(12) = \frac{1}{36}$.

Solution for exercise 1.3

Probabilities for $\mathcal{S}_A$: $P_A(1) = \cdots = P_A(4) = \frac{5}{24}$ and $P_A(5) = P_A(6) = \frac{1}{12}$.

Probabilities for $\mathcal{S}_B$: $P_B(t1) = \cdots = P_B(t6) = \frac{1}{12}$ and $P_B(h1) = \cdots = P_B(h4) = \frac{1}{8}$.

$P_A(3) + P_A(5) = \frac{7}{24}$.

$P_B(t3) + P_B(t5) + P_B(h3) = \frac{7}{24}$.

Solution for exercise 1.4

Solution for exercise 1.5

$P(t) = \frac{2}{3}$.

$P(4 \,|\, t) = \frac{1}{12}$.

$P(4 \,|\, t) + P(5 \,|\, t) + P(6 \,|\, t) = \frac{1}{4}$.

$P(4 \,|\, h) = \frac{3}{8}$.

The four-sided die thus has a higher probability of rolling 4 or more, than the six-sided die.

Solution for exercise 1.6

Starting with (equation just before chap 1.2.2) we get

$$\begin{aligned}
P(\mathcal{A} \,|\, \mathcal{B} \cap \mathcal{C}) &= \frac{P(\mathcal{A} \cap \mathcal{B} \cap \mathcal{C})}{P(\mathcal{B} \cap \mathcal{C})} \\
\Leftrightarrow P(\mathcal{A} \,|\, \mathcal{B} \cap \mathcal{C}) &= \frac{P(\mathcal{A} \cap \mathcal{B} \cap \mathcal{C})}{P(\mathcal{B} \,|\, \mathcal{C}) P(\mathcal{C})} \\
\Leftrightarrow P(\mathcal{A} \,|\, \mathcal{B} \cap \mathcal{C}) P(\mathcal{B} \,|\, \mathcal{C}) &= \frac{P(\mathcal{A} \cap \mathcal{B} \cap \mathcal{C})}{P(\mathcal{C})} \\
\Leftrightarrow P(\mathcal{A} \,|\, \mathcal{B} \cap \mathcal{C}) P(\mathcal{B} \,|\, \mathcal{C}) &= \mathcal{P}(\mathcal{A} \cap \mathcal{B} \,|\, \mathcal{C}) \ .
\end{aligned}$$

Here the first step is application of the fundamental rule, and the last step is exploitation of the definition of conditional probability.

Solution for exercise 1.7

We have (p stands for “pregnant” and t for “test is positive”)

$$\begin{aligned}
P(1) &= P(p) P(t \,|\, p) = 0.05 \cdot 0.98 = 0.049 \\
P(2) &= P(p) P(\neg t \,|\, p) = 0.05 \cdot 0.02 = 0.001 \\
P(3) &= P(\neg p) P(t \,|\, \neg p) = 0.95 \cdot 0.001 = 0.00095 \\
P(4) &= P(\neg p) P(\neg t \,|\, \neg p) = 0.95 \cdot 0.999 = 094905 \ .
\end{aligned}$$

$P(p \,|\, t) = \frac{P(1)}{P(1)+P(3)} = 0.981$.

Solution for exercise 1.8

Sample space (r stands for "red", and b for "blue"):

$$\mathcal{S} = \{r1b1, r1b2, \ldots, r1b6, r2b1, r2b2, \ldots, r6b6\} .$$

$P(r1b1) = P(r1b2) = P(r1b3) = P(r2b1) = \cdot = P(r6b3) = \frac{1}{6} \cdot \frac{1}{12} = \frac{1}{72}$.

$P(r1b4) = P(r1b5) = P(r1b6) = P(r2b4) = \cdot = P(r6b6) = \frac{1}{6} \cdot \frac{1}{4} = \frac{1}{24}$.

Solution for exercise 1.9

We have the two variables C and D, with C having states h and t, and D having states 1 to 6. We have $P_C(h) = \frac{1}{3}$ and $P_C(t) = \frac{2}{3}$. We have $P_D(1) = \frac{5}{18} + \frac{1}{24} = \frac{23}{72}$, $P_D(2) = \frac{1}{9} + \frac{1}{24} = \frac{11}{72}$, $P_D(3) = \frac{1}{9} + \frac{1}{8} = \frac{17}{72}$, $P_D(4) = \frac{1}{18} + \frac{1}{8} = \frac{13}{72}$, and $P_D(5) = P_D(6) = \frac{1}{18}$.

Solution for exercise 1.10

Define

$$\begin{aligned}
\mathcal{C}_a &= \{(a', b') \in \mathrm{sp}(A) \times \mathrm{sp}(B) \ : \ a' = a\}, \\
\mathcal{C}_b &= \{(a', b') \in \mathrm{sp}(A) \times \mathrm{sp}(B) \ : \ b' = b\}, \\
\mathcal{C}_{ab} &= \{(a', b') \in \mathrm{sp}(A) \times \mathrm{sp}(B) \ : \ a' = a \text{ and } b' = b\}.
\end{aligned}$$

Obviously $\mathcal{C}_{ab} = \mathcal{C}_a \cap \mathcal{C}_b$. We thus have

$$\begin{aligned}
& P(\mathcal{C}_{ab}) = P(\mathcal{C}_a \cap \mathcal{C}_b) \\
\Leftrightarrow & \frac{P(\mathcal{C}_{ab})}{P(\mathcal{C}_b)} = \frac{P(\mathcal{C}_a \cap \mathcal{C}_b)}{P(\mathcal{C}_b)} \\
\Leftrightarrow & \frac{P(\mathcal{C}_{ab})}{P(\mathcal{C}_b)} = P(\mathcal{C}_a \,|\, \mathcal{C}_b) \\
\Leftrightarrow & P(\mathcal{C}_{ab}) = P(\mathcal{C}_a \,|\, \mathcal{C}_b) P(\mathcal{C}_b).
\end{aligned}$$

As this is valid for any choice of states a and b, the fundamental rule holds for variables.

Solution for exercise 1.11

$P(A) = (0.2, 0.4, 0.4), P(B) = (0.3, 0.3, 0.4)$.
$P(A|b_1) = (0.167, 0.5, 0.333), P(A|b_2) = (0.333, 0, 0.667), P(A|b_3) = 0.125, 0.625, 0.25)$.
$P(B|a_1) = (0.25, 0.5, 0.25), P(B|a_2) = (0.375, 0, 0.625), P(B|a_3) = (0.25, 0.5, 0.25)$.

Solution for exercise 1.12

(i) 0.996

(ii) 0.498

Solution for exercise 1.13

(i) $P(B, c_1) = (0.02, 0.08), P(B, c_2) = (0.18, 0.72), P(B) = (0.2, 0.8)$

(ii) $P(A|b_1, c_1) = (0.3, 0.7) = P(A|b_1, c_2), P(A|b_2, c_1) = (0.6, 0.4) = P(A|b_2, c_2)$.

Solution for exercise 1.14

Solution for exercise 1.15

The associative, commutative, and distributive laws all follow from Property 1 of potentials, that multiplication of potentials is defined to be pointwise, and that standard multiplication and summation follows the associative, commutative, and distributive law. We therefore only show the most involved case, namely the distributive law, in details:

For the distributive law, let $\mathcal{C}_{12} = \mathrm{dom}\,(\phi_1) \cap \mathrm{dom}\,(\phi_2)$, $\mathcal{C}_1 = \mathrm{dom}\,(\phi_1) \setminus \mathcal{C}_{12}$, and $\mathcal{C}_2 = \mathrm{dom}\,(\phi_2) \setminus (\mathcal{C}_{12} \cup \{A\})$. Obviously the sets are disjoint and $\mathcal{C}_1 \cup \mathcal{C}_{12} \cup \mathcal{C}_2 \cup \{A\} = \mathrm{dom}\,(\phi_1\phi_2)$. Let $c_{12} \in \mathrm{sp}(\mathcal{C}_12)$, $c_1 \in \mathrm{sp}(\mathcal{C}_1)$, and $c_2 \in \mathrm{sp}(\mathcal{C}_2)$. We calculate

$$\begin{aligned}
(\sum_A \phi_1\phi_2)(c_1, c_{12}, c_2) &= \sum_{a \in \mathrm{sp}(A)} (\phi_1\phi_2)(c_1, c_{12}, c_2, a) \\
&= \sum_{a \in \mathrm{sp}(A)} \phi_1(c_1, c_{12}, c_2, a)\phi_2(c_1, c_{12}, c_2, a) \\
&= \sum_{a \in \mathrm{sp}(A)} \phi_1(c_1, c_{12})\phi_2(c_{12}, c_2, a) \\
&= \phi_1(c_1, c_{12}) \sum_{a \in \mathrm{sp}(A)} \phi_2(c_{12}, c_2, a) \ .
\end{aligned}$$

As c_1, c_{12}, and c_2 were selected arbitrarily the result follows.

Solution for exercise 1.16

Solution for exercise 1.17